Feed-forward and its role in conditional linear optical quantum dynamics

S. Scheel,[1,*] W.J. Munro,[2] J. Eisert,[1,3] K. Nemoto,[4] and P. Kok[2]

[1]*Quantum Optics and Laser Science, Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BW, UK*
[2]*Hewlett-Packard Laboratories, Filton Road, Stoke Gifford, Bristol BS34 8QZ, UK*
[3]*Institute for Mathematical Sciences, Imperial College London, Prince's Gardens, London SW7 2PE, UK*
[4]*National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan*

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Nonlinear optical quantum gates can be created probabilistically using only single photon sources, linear optical elements and photon-number resolving detectors. These gates are heralded but operate with probabilities much less than one. There is currently a large gap between the performance of the known circuits and the established upper bounds on their success probabilities. One possibility for increasing the probability of success of such gates is feed-forward, where one attempts to correct certain failure events that occurred in the gate's operation. In this brief report we examine the role of feed-forward in improving the success probability. In particular, for the non-linear sign shift gate, we find that in a three-mode implementation with a single round of feed-forward the optimal average probability of success is approximately given by $p_{\text{success}} = 0.272$. This value is only slightly larger than the general optimal success probability without feed-forward, $p_{\text{success}} = 0.25$.

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I. INTRODUCTION

In recent years we have seen the emergence of photonic states of light as a possible medium for achieving universal quantum computation and the medium of choice for quantum communication. Many of the photon's properties, such as its clean manipulation and negligible decoherence, makes it ideal to achieve this goal. However, for scalable quantum computing we require photons to interact with one another. Without this interaction any computation could be efficiently simulated classically. To achieve such interactions it was thought that massive reversible nonlinearities were required [1]. However, materials giving such large nonlinearities are well beyond our ability to manufacture. Then, the pioneering work of Knill, Laflamme, and Milburn (KLM) [2] showed that with only single-photon sources, photon-number resolving detectors, linear optical elements such as beam splitters, and feed-forward of measurement outcomes, a near-deterministic controlled-NOT (CNOT) gate could be created based on the so-called dual-rail encoding. This procedure uses a very significant fixed overhead of resources, using "ancilla systems", to achieve some overall failure rate, e.g., one below the threshold for fault-tolerant quantum computation. With this architecture for the CNOT gate and single-qubit rotations – accessible with linear optics – a universal set of gates was possible and a route forward for creating large devices can be seen Note that here universal means that an operation can be implemented that approximates the representation of any unitary to arbitrary accuracy, based on the dual-rail encoding, where the logical qubit is encoded in $|1,0\rangle$ and $|0,1\rangle$. Hence, it does not mean that any physical gate, as the nonlinear sign shift gate, can be realized with any probability of success. Since this original work there has been significant progress both theoretically [3, 4, 5, 6, 7, 8, 9] and experimentally [10, 11, 12], with a number of CNOT gates actually having been demonstrated.

Much of the theoretical effort has focused on determining more efficient ways to perform the controlled logic. There are two building blocks of particular importance, the first being the already mentioned controlled-NOT gate and the second being the so-called nonlinear sign-shift (NS) gate. This second gate takes a general two-photon state composed of a superposition of number states with zero, one, and two photons and flips the sign of the $|2\rangle$ component. So it acts as

$$c_0|0\rangle + c_1|1\rangle + c_2|2\rangle \longmapsto c_0|0\rangle + c_1|1\rangle - c_2|2\rangle, \qquad (1)$$

where $|n\rangle$ is the n-th number state vector of the optical field and the coefficients satisfy the usual normalization constraint. The NS gate is of interest because it is technically more primitive and fundamental than the CNOT gate, in fact two NS gates (in addition to two Hadamard gates) can be used to construct a CNOT gate. Using the standard models of linear optical logic it has been shown in Ref. [5] that the maximum probability for achieving the NS gate with postselection is bounded from above by $1/2$ (and $3/4$ for the CNOT gate). These upper bounds are known not to be tight, but they already indicate that near-deterministic gates are not possible using only the linear optical resources, toolbox, and strategy. Note again that this is no contradiction: using the KLM scheme, these non-linear gates can not be implemented with an arbitrary probability of success, but in turn only the representation of any unitary in terms of the dual-rail encoding, taken as the encoding of the logical qubits.

It has been shown for small photon numbers in the ancilla system [13, 14] and later in generality [15] that without feed-forward operations (operations that correct situations in which the gate has not irrecoverably failed) or the use of nonlinear optical resources the maximum probability of success with unlimited ancilla is only $1/4$. Ironically, this is just the value attained in the original proposal in Ref. [2], so this bound is tight. This still leaves a lot of space for improvement with po-

*Electronic address: s.scheel@imperial.ac.uk

tential appropriate feed-forward steps already on the level of NS gates – with significant implications on the required overhead in resources in the scalable scheme. It is manifest that this very significant overhead in resources in the full scheme including feed-forward is indeed dictated by the success probability of the elementary NS gate.

In this article, we will investigate the possibility of raising the success probability of the NS gate using feed-forward steps already on the level of elementary gates, and not only in the full scalable scheme [19]. Although we consider only restricted settings taking the minimal number of auxiliary modes into account, the findings suggest that even with feed-forward and correction steps, the success probability cannot be uplifted much at all.

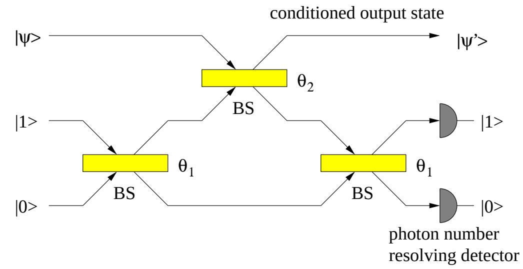

FIG. 1: Schematic diagram of the original KLM nonlinear sign shift gate. The three input states are the unknown two photon signal state vector $|\psi\rangle$, plus two ancilla modes, one initially prepared as a single photon $|1\rangle$ and the second as a vacuum $|0\rangle$. They interact with each other via the beam splitters characterized by $\theta_1, \theta_2 \in [0, 2\pi)$, respectively. Here, $\cos^2\theta_1 = 1/(4-2\sqrt{2})$ and $\cos^2\theta_2 = 3-2\sqrt{2}$. The ancilla modes are then measured using photon-number resolving detectors. Upon obtaining the pattern $|1,0\rangle$, the signal state vector $|\psi\rangle$ is transformed to $|\psi'\rangle = c_0|0\rangle + c_1|1\rangle - c_2|2\rangle$.

II. THE NONLINEAR SIGN-SHIFT GATE

The simplest nonlinear operation/network to be constructed with linear optical techniques is the nonlinear sign shift gate originally proposed by KLM. An implementation of the gate is depicted in Fig. 1, involving the signal mode and two auxiliary modes. The linear optics network is in this simple three-mode set-up characterized by two angles $\theta_1, \theta_2 \in [0, 2\pi)$, with $\cos\theta_{1,2}$ denoting the beam splitter transmittivities. It is straightforward and illustrative to show that for an initial signal mode input $|n\rangle$ the above gate, conditioned on the $|1,0\rangle$ detection pattern of the detectors acting on the two auxiliary modes, with arbitrary angles θ_1 and θ_2, yields a transformation as [2]

$$\begin{aligned}
|0\rangle &\mapsto \left[\cos^2\theta_1\cos\theta_2 + \sin^2\theta_1\right]|0\rangle, \\
|1\rangle &\mapsto -\left[\cos^2\theta_1\cos 2\theta_2 + \sin^2\theta_1\cos\theta_2\right]|1\rangle, \qquad (2)\\
|2\rangle &\mapsto \cos\theta_2\left[\frac{1}{2}\cos^2\theta_1(1-3\cos 2\theta_2) - \sin^2\theta_1\cos\theta_2\right]|2\rangle.
\end{aligned}$$

Now with θ_1 and θ_2 chosen such that $\cos^2\theta_1 = 1/(4-2\sqrt{2})$ and $\cos^2\theta_2 = 3-2\sqrt{2}$, all these three transformations have the same amplitude of $1/2$ with the $|2\rangle$-component also having a negative sign. Hence, a general two-photon signal state vector $|\psi\rangle$ is transformed according to

$$\begin{aligned}
|\psi\rangle &= c_0|0\rangle + c_1|1\rangle + c_2|2\rangle \\
&\longmapsto \frac{1}{2}|\psi'\rangle = \frac{1}{2}\left[c_0|0\rangle + c_1|1\rangle - c_2|2\rangle\right]. \qquad (3)
\end{aligned}$$

The loss in amplitude reflects the existence of other measurement outcomes, and so this heralded transformation is effected with a success probability of 1/4.

III. BOUND ON SUCCESS PROBABILITIES

Let us first examine the limits on the maximum probability of success. Firstly, for the NS gate Knill [5] has established a loose upper bound of $1/2$ using a photon-number conservation argument ($1/4$ is a tight bound without feed-forward [15]). We will investigate the potential for improvement by examining the unsuccessful outcomes of the gate in a three-mode implementation, that is, the situations where we do not measure $|1,0\rangle$. These unsuccessful outcomes fit into three categories: i) Detection of two or more photons in all the ancilla modes. In these cases some information in the signal state is irreversibly destroyed because of photon subtraction. ii) Detection of $|0,1\rangle$. In this case the photon number in the signal mode has not changed but an incorrect transformation has been applied. iii) Detection of $|0,0\rangle$. In this case a photon has been added to the signal mode and an incorrect transformation has been applied.

The first category contains outcomes where more than one photon is detected. Consequently, more than zero photons must have been present in the signal state, and we have therefore obtained information about the state. Such an error is irrecoverable, as information has leaked out of the system. The irrecoverable errors are two-photon patterns – $|1,1\rangle$, $|2,0\rangle$, and $|0,2\rangle$ – and three-photon patterns – $|2,1\rangle$, $|1,2\rangle$, $|3,0\rangle$, and $|0,3\rangle$. We need to calculate the probabilities of these patterns, as their sum gives the total irrecoverable error probability. This is an indication of the maximum upper probability bound for the gate to work. The maximum probability of success for the gate must be less than one minus this irrecoverable error probability.

To determine the error probabilities we use the techniques introduced in Ref. [18] to calculate all the necessary matrix elements of any unitary $\hat{U}$ acting in state space as

$$\begin{aligned}
\langle m_1, m_2, m_3|\hat{U}|n_1, n_2, n_3\rangle &= \left(\prod_{i,j} m_i! n_j!\right)^{-1/2} \\
&\times \text{per}\,\mathbf{\Lambda}[(1^{m_1}, 2^{m_2}, 3^{m_3})|(1^{n_1}, 2^{n_2}, 3^{n_3})]. \qquad (4)
\end{aligned}$$

Here, the multi-index $(1^{m_1}, 2^{m_2}, 3^{m_3})$ corresponds to an index collection in which the index i occurs m_i times. The symbol "per" denotes the *permanent* of the unitary $\mathbf{\Lambda}$ acting on the bosonic annihilation operators associated with the unitary $\hat{U}$.

The probability of getting one of the wrong results is state-dependent, since the corresponding transformation does not

constitute a unitary operation on the signal state. There are then several different ways of proceeding: One could calculate the *average* failure rate by averaging the failure probability over all possible input states. In this case we obtain

$$\bar{p}_{\text{failure}} = \frac{41}{\sqrt{2}} - \frac{86}{3} \approx 0.325\,. \tag{5}$$

On the other hand, one can calculate the *maximal* failure rate by looking at the class of input state for which the failure probability becomes extremal. This is the case when $c_0 = c_1 = 0$ and $c_2 = 1$ in which case we obtain

$$p_{\text{max,failure}} = 57\sqrt{2} - 80 \approx 0.61\,. \tag{6}$$

This failure probability is larger than the suggested maximal failure rate of $1/2$ in Ref. [5] which hints at a possible strictly lower bound on the success probability than $1/2$. It is more adequate to consider the maximal failure rate as it gives truly the worst performance of the gate which is more appropriate when setting bounds [20]. Obviously, the (state-independent) success probability cannot be larger than one minus the (state-dependent) maximal failure rate. So far in our considerations we have looked only at the irrecoverable errors. There are two other ancilla detection patterns ($|0,1\rangle$ and $|0,0\rangle$) which correspond to incorrect transformations that do not destroy the information in the signal state.

IV. CORRECTABLE ERROR EVENTS

We will briefly look at the cases in which the measurement pattern does not result in a complete failure, but in a potentially recoverable error. For simplicity we will assume for the moment that the network has been tuned to produce the maximal success rate, a condition that will be relaxed later. Let us consider first the situation in which no photons are detected in the ancilla modes, that is, our measurement result $|0,0\rangle$ occurs from the ancilla $|1,0\rangle$ input. In this case the (unnormalized) output state vector from the NS gate is

$$\begin{aligned} c_0 2^{-1/4}|1\rangle \;&+\; c_1 2^{1/4}(1-\sqrt{2})|2\rangle \\ &+\; c_2 2^{-1/4}(51-36\sqrt{2})^{1/2}|3\rangle\,. \end{aligned} \tag{7}$$

That is, the information about the input state is still there, but the state contains too many photons. The smallest term in this equation has an amplitude of $2^{-1/4}(51-36\sqrt{2})^{1/2} \approx 0.25$ and so can at best only increase the success probability for the worst input by approximately $0.25^2 = 0.062$ to $0.25 + 0.062 = 0.312$. To achieve that, one has to assume the possibility of perfect recovery, which is unlikely. Hence, let us determine how efficiently we can recover from this error syndrome. This will be achieved by applying a second conditional network that subtracts one photon. There are several possibilities available to us. The simplest one would be a single beam splitter with a vacuum input and a single-photon detection. That, however, does not contain enough parameters to enable the correction to occur with nonzero probability.

Similarly, an SU(3) network (depicted in Fig. 2) with $|0,0\rangle$-ancilla and $|1,0\rangle$-detection does not help since only two of the

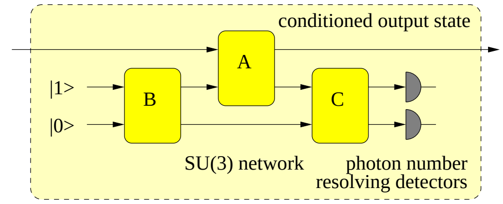

FIG. 2: A schematic diagram of an SU(3)-network with an initial $|10\rangle$-ancilla. The boxes A, B and C represent general SU(2)-networks.

beam splitters (A and C in Fig. 2) effectively take part in the process. That means, that at least one photon has to take part in the recovery process and so we must start with an ancilla of the type $|1,0\rangle$ again. For that we have to make a choice of the measurement pattern again. If we choose $|1,1\rangle$, then with the beam splitter angle choices $\theta_A = \theta_C = 0.489377$, $\theta_B = 1.07621$ we end up with a success probability of only approx. 0.007 which increases the overall probability of success of our gate to 0.257. This improvement is very small and rather discouraging.

However, there is one more error syndrome we can attempt to correct. This is the situation in which our measurement result was $|0,1\rangle$. The number of photons in the probe beam has not changed but the two-photon input state vector has been transformed into

$$\frac{1-\sqrt{2}}{2}c_0|0\rangle + \frac{5-3\sqrt{2}}{2}c_1|1\rangle + \frac{15-11\sqrt{2}}{2}c_2|2\rangle\,. \tag{8}$$

The smallest amplitude in this state has an amplitude of $\left|(1-\sqrt{2})/2\right| \approx 0.21$ and thus corresponds to a probability of 0.043. Again, it seems unlikely we can achieve this total correction by linear optical techniques. Using our general SU(3) network with $\theta_A = \theta_C = 2.53787$, $\theta_B = 2.26111$ we can correct this error syndrome with a total probability of 0.015 (compared to the maximal possible value of 0.043). Now adding all the successful error-syndrome-corrected probabilities together with our $1/4$ initial gate success probability we get

$$p_{\text{total success}} \approx 0.272. \tag{9}$$

This is a slight improvement but far less than the upper bound of $1/2$, or even the upper bound $1-0.61 = 0.39$ from Eq. (6). In fact, if we were to take the stance that we could somehow correct all the recoverable error syndromes, then we would have had a success probability $p_{\text{maximal success}} \approx 0.355$.

So far we have looked only at a single round of error syndrome correction. We can of course attempt to correct the recoverable errors from the first round (depicted in Fig. 3). Unfortunately, a numerical study provides some evidence that this only changes the total success probability $p_{\text{total success}}$ by less than one percent and also decreases $p_{\text{maximal success}}$ by several percent.

The result so far is that one or two recovery steps using additional SU(3) networks do not greatly help to improve the success probability of the NS gate. We have assumed until now that the first network has been individually optimized

with respect to its probability of success which seems to be a sensible thing to do experimentally. Let us now relax this condition. Now we consider in the first instance two SU(3) networks as in Fig. 3 with three beam splitters each. Using Eq. (4) with the 3×3 unitary $\mathbf{\Lambda}$ being a concatenation of the beam splitter matrices with angles θ_A, θ_B, $\theta_C \in [0, 2\pi)$, respectively, we obtain for the matrix elements of $\hat{U}$

$$\begin{aligned}
\langle 0,1,0|\hat{U}|0,1,0\rangle &= \text{per}\,\mathbf{\Lambda}[2|2] = \Lambda_{22}\,, \qquad (10)\\
\langle 1,1,0|\hat{U}|1,1,0\rangle &= \text{per}\,\mathbf{\Lambda}[1,2|1,2] = \Lambda_{11}\Lambda_{22} + \Lambda_{12}\Lambda_{21}\,,\\
\langle 2,1,0|\hat{U}|2,1,0\rangle &= \text{per}\,\mathbf{\Lambda}[1,1,2|1,1,2]\\
&= \Lambda_{11}(\Lambda_{11}\Lambda_{22} + \Lambda_{12}\Lambda_{21} + \Lambda_{11}\Lambda_{12})\,.
\end{aligned}$$

Note that in each network two beam splitter angles are constrained by the requirement of performing a particular gate operation whereas the third angle determines the probability of success. Now we maximize the total success probability of the concatenated networks by optimizing the beam splitter angles in both networks simultaneously, rather than individually. It is worth noting that the success probabilities of ob-

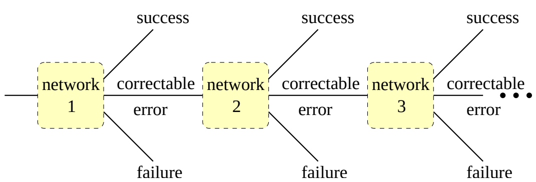

FIG. 3: Sequence of conditional networks. Each application can have one of three possible outcomes: success, correctable error, or uncorrectable failure.

taining the outcomes $|1,0\rangle$ and $|0,1\rangle$ after the first network behave exactly oppositely. That is, as a function of the third beam splitter angle, the minimum in the probability of finding the measurement result $|0,1\rangle$ occurs exactly where the probability of finding $|1,0\rangle$ is maximal. This in turn means that there is a balance between succeeding with the first network and failure recovery with a subsequent network. If we denote by $p_x^{(i)}$ the probability of obtaining the measurement outcome x in the i-th network, numerical optimization yields for the overall probability

$$p_{\text{total success}} = p_{1,0}^{(1)} + p_{0,1}^{(1)} p_{1,0}^{(2)} \approx 0.28\,. \qquad (11)$$

Again, this is very low, and suggests that there should be a tighter bound than $1/2$ for single rounds of feed-forward. Moreover, the maximal failure probability turns out to be $p_{\text{max,failure}} \approx 0.66$, so that there is only a chance of at most $1/3$ to generate the nonlinear sign shift gate even with more subsequent conditional networks.

V. CONCLUSIONS

We have shown that it is possible to construct nonlinear sign shift gates that have a probability of success exceeding $1/4$ with linear optics, photo-detection and feed-forward. These techniques are applicable when probabilistic gates fail without divulging information about the input state. This recycling or syndrome recovery allows a very modest increase in the overall success probability from $1/4$ to approx. 0.28. This suggests that the success probability of the building blocks can not be significantly reduced by introducing single feed-forward steps already on the level of the building blocks, with the implication of a not very large reduction of the overhead in resources in the full scalable scheme, in turn making use of feed-forward. Multiple SU(3) networks will be required consuming a significant number of signal photons. While these resources are constant for a fixed overall success probability, they are daunting for an experimentalist. While our techniques have been applied directly to the NS gate, a similar analysis can be applied to the other conditional linear optical gates such as the CNOT gate.

Our results suggests that it is not practical to use feed-forward operations to correct the error syndromes where no information has been erased about our quantum states or processes. If we want to significantly improve the success probability of these elementary gates, we need to move outside the linear optical toolbox by, for example, utilizing other sources of nonlinearity.

Acknowledgments

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[19] Note that it has been recently established that by expanding the linear optical set to include (weak) nonlinearities one can construct near-deterministic controlled gates using homodyne measurements [16] or the quantum Zeno effect [17].
[20] In the same spirit, we can examine the situation in which we have an initial $|1,1\rangle$-ancilla rather than $|1,0\rangle$. In this situation the maximum probability of success for the gate is 0.236 with a maximal failure probability of $p_{\text{max,failure}} \approx 0.546$ which again exceeds $1/2$.